Calibrating Stochastic Models with Bayesian Inference

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There are two different schools of thought on the concept of statistics and probability:

- in the frequentist approach, probabilities are fixed and represent the frequency of an event that will happen in a large number of observations, while
- in the Bayesian approach, probabilities are interpreted as one's belief in the plausibility of an outcome from occurring, and this belief is constantly updated upon the arrival of new information.

Using the housing market to illustrate these differences, frequentists would calculate the probability of foreclosure in a given area using its long-run historical data and assume that this probability is fixed; while the Bayesian approach would adjust this probability dynamically based on latest observations of various factors such as the underlying economic conditions, interest rates and credit scores.

Although frequentist statistics is widely used in various applications, critics often highlight the rigid approach as its Achilles' heel. In fact, the 2008 financial crisis can be attributed to the use of frequentist probabilities, where the assumption of a fixed foreclosure rate was one of the reasons that led to the understatement of risks. Given the subjective nature and large uncertainty in quantitative finance, the Bayesian approach is now the preferred method because it overcomes these limitations by:

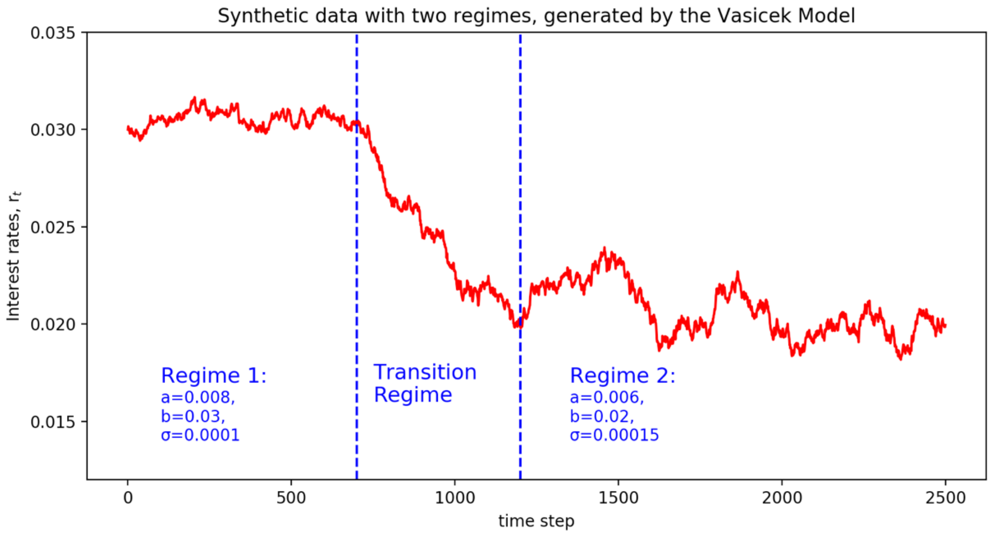
- providing a natural framework of incorporating prior information with new observations, and also
- accounting for uncertainties.

In this article, **we briefly describe a novel methodology that uses Bayesian learning to calibrate a stochastic model**, in particular, the Vasicek model that is commonly used to simulate mean-reverting patterns such as the movement of interest rates,

$$dr_t = a(b - r_t)dt + \sigma dW_t,$$

where r_t is the interest rate at time t, dr_t is the change in interest rate at time t, a is the speed of mean reversion, b is the long-term mean value of r_t, σ is the volatility size, and dW_t is the Wiener process that uses a standard normal distribution to describe a random 1D Brownian motion. In other words, the Vasicek model assumes that interest rates fluctuate around the long-term mean b with an amplitude of σ, and a determines the speed at which r_t reverts back to b.

The figure below shows the movement of interest rates that is generated using a Vasicek model with two different set of parameters (θ_{Vasicek}: a, b and σ). The first set of data points (time step, $t < 700$), labelled "Regime 1", were generated with a mean-reversion speed (a) of 0.008 with an equilibrium value (b) of 0.03 and a volatility (σ) of 0.0001; while the final set of data points ($t > 1200$), "Regime 2", were produced using a lower mean-reversion speed ($a = 0.006$) and equilibrium value ($b = 0.02$) with a larger volatility ($\sigma = 0.00015$). Here, we show that a Bayesian inference algorithm is capable of extracting the evolution of θ_{Vasicek} by observing the change in interest rates at each time step.



The Bayesian inference algorithm requires an initial estimate of the model parameters (θ_{Vasicek}: a, b and σ) and its search space to be defined. We fit the first 10% of the data points ($r_{0:250}$) with a maximum likelihood estimate (MLE) and defined the search space of each θ_{Vasicek} to be between 10% and 200% of their initial MLE estimates. While the search space is fixed in this example, the algorithm can easily be extended to account for a dynamic search space.

With the initial estimate and search space defined, we can now use a Bayesian approach to estimate the possible range of θ_{Vasicek} for each time step,

$$p(\theta \mid r_t, dr_t) = \frac{p(r_t, dr_t \mid \theta)\, p(\theta)}{\sum p(r_t, dr_t \mid \theta)\, p(\theta)}.$$

The prior, $p(\theta)$, represents the probability of selecting a given combination of model parameters θ_{Vasicek}, and we divide the search space of each θ_{Vasicek} by 100 intervals of equal width ($n_{\text{bins}} = 100$) to initialize the prior, $p(\theta^{\text{Initial}})$. For each time step t, the data points (r_t and dr_t) are fed into the algorithm where $p(r_t, dr_t \mid \theta)$, the probability of observing r_t and dr_t for all possible combinations of θ_{Vasicek}, is calculated. $p(\theta)$ and $p(r_t, dr_t \mid \theta)$ are then used to calculate the posterior, $p(\theta \mid r_t, dr_t)$, where our beliefs of the underlying θ_{Vasicek} are updated based on the arrival of new information (r_t and dr_t). At the end of each time step, we calculate

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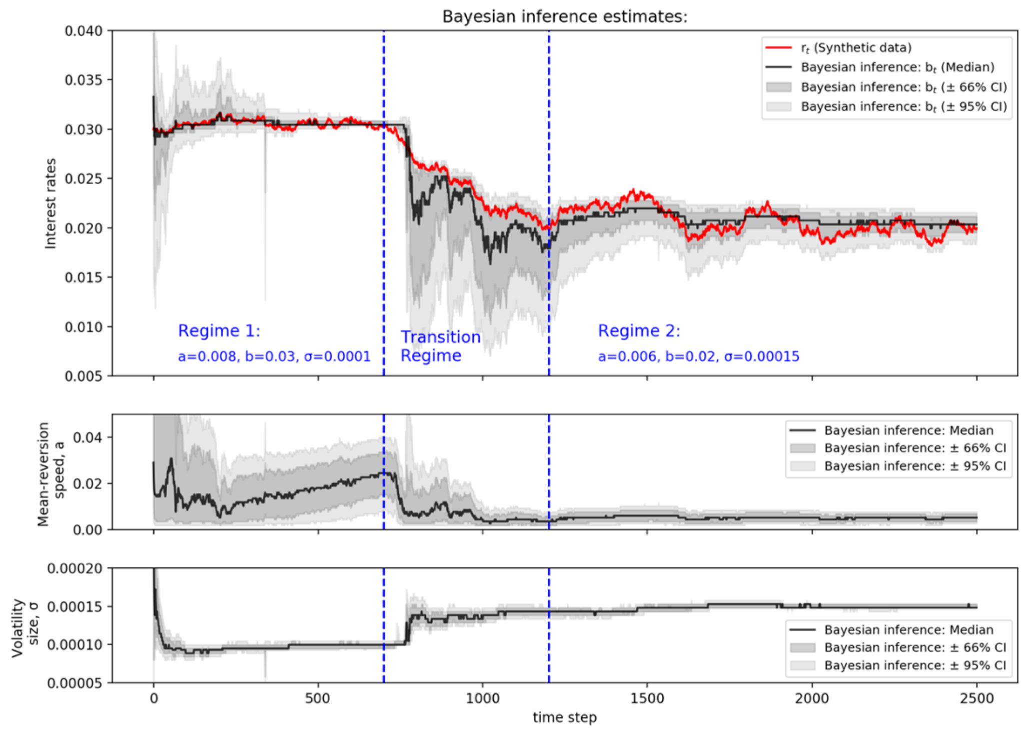
the confidence interval of each θ_{Vasicek} and update our priors as a weighting between $p(\theta \mid r_t, dr_t)$ and $p(\theta^{\text{Initial}})$,

$$p(\theta) = \omega_{\text{belief}}\, p(\theta \mid r_t, dr_t) + (1 - \omega_{\text{belief}})\, p(\theta^{\text{Initial}}),$$

where ω_{belief} is the belief discount factor that ranges between 0 (larger emphasis on past data) and 1 (lower emphasis). This is used to add uncertainty in our beliefs in θ_{Vasicek} by flattening the prior distribution, where we specify ω_{belief} to be 0.9999.

An estimate of θ_{Vasicek} (a, b and σ) and their uncertainty bounds for each time step is shown in the figure below. The estimated θ_{Vasicek} (from the Bayesian inference algorithm) is in good agreement with the underlying θ_{Vasicek} that were used to generate the underlying interest rates, r_t (red line). Uncertainty in the estimated equilibrium value (b_t) decreases when the underlying interest rate (r_t) is stable and when more observations are processed by the algorithm. The estimated θ_{Vasicek} also correctly adjusts to the change in regime, with larger uncertainties at the transition regime.



Unlike frequentist approaches that only provide single point estimate for a given dataset, we have shown that a Bayesian approach can be superior because it:

- allows us to dynamically update our beliefs upon the arrival of new information, and
- provides a confidence interval to reflect uncertainty in the estimated quantities.

The Bayesian approach is widely used in machine learning applications and can also be easily applied to calibrate other stochastic models such as a geometric Brownian motion.

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